Algorithmic Graph Theory

Winter 2014/15

6. Lecture

Rooted Spanning Trees
Rooted Trees

**Def.** A digraph $T = (V, E)$ with node $s \in V$ is called an *$s$-rooted tree*, if

- $T$ is acyclic,
- $\text{indeg}(s) = 0$ and
- $\text{indeg}(v) = 1$ for every node $v \in V - s$. 
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$T$ contains no (directed) cyclic
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![Diagram of rooted trees](image-url)
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![Diagrams](image-url)
Def. Let $G = (V, E)$ be a (multi-) digraph with node $s \in V$. A subgraph $T$ of $G$ with node set $V$ is called $s$-rooted spanning tree of $G$, if $T$ is a $s$-rooted tree.
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Existence of Rooted Spanning Trees

**Obs.** Let $G$ be a (multi-) digraph with node $s$. $G$ contains an $s$-rooted spanning tree, if any only if every node $v \in V$ is reachable from $s$ in $G$. 
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ex. $s$-$v$ path in $G$
Existence of Rooted Spanning Trees

**Obs.** Let $G$ be a (multi-) digraph with node $s$. $G$ contains an $s$-rooted spanning tree, if and only if every node $v \in V$ is reachable from $s$ in $G$.

**Proof.**
See exercise sheet.
Existence of Rooted Spanning Trees

**Obs.** Let $G$ be a (multi-) digraph with node $s$. $G$ contains an $s$-rooted spanning tree, if any only if every node $v \in V$ is reachable from $s$ in $G$.

**Proof.**

See exercise sheet.

**Remark.**

DFS($s$) gives $s$-rooted spanning tree (if there is any).
Minimum Rooted Spanning Trees

**Def.** Given: (multi-) digraph $G = (V, E)$ with $s \in V$ and edge costs $c : E \to \mathbb{R}_{\geq 0}$. 
Minimum Rooted Spanning Trees

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Task: Find $s$-rooted spanning tree $T = (V, E_T)$ of $G$ (if there is any) of minimum cost $c(E_T) = \sum_{e \in E_T} c(e)$. 
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**Motivation:**
Broadcast (send information from $s$ to all other nodes) in a communication network.
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**Exercise:**
Kruskal and Jarník-Prim fail in general!
Cost Modification

For every $v \neq s$ set $c_0(v) := \min\{ c(u, v) | (u, v) \in E \}$
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W.l.o.g. indeg($v$) $\geq 1$ for every $v \neq s$. 
Cost Modification

For every $v \neq s$ set $c_0(v) := \min\{ c(u, v) \mid (u, v) \in E \}$

W.l.o.g. $\text{indeg}(v) \geq 1$ for every $v \neq s$.

For every edge $(u, v)$ in $G$ set $c'(u, v) := c(u, v) - c_0(v) \geq 0$. 

![Diagram of graph transformation]
Cost Modification

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W.l.o.g. \( \text{indeg}(v) \geq 1 \) for every \( v \neq s \).

For every edge \( (u, v) \) in \( G \) set \( c'(u, v) := c(u, v) - c_0(v) \geq 0. \)

\( \Rightarrow \) every node \( v \neq s \) has incoming 0-edge.
Validity of the Cost Modification

Lem\textsuperscript{1}. An $s$-rooted spanning tree of $G$ is optimum with respect to $c$, if and only if it is optimum with respect to $c'$. 
Validity of the Cost Modification

**Lem**\(^1\). An \(s\)-rooted spanning tree of \(G\) is optimum with respect to \(c\), if and only if it is optimum with respect to \(c'\).

**Proof.**

For every \(s\)-rooted spanning tree \(T = (V, E_T)\) we have
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\(\text{indeg}_T(v) = 1\) for all \(v \neq s\) and \(\text{indeg}_T(s) = 0\)
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= c(E_T) - \sum_{v \in V-s} c_0(v)
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\(\square\)
An Attempt

Pic for every \( v \neq s \) an incoming 0-Kante \( \sim \) subgraph \( F = (V, E_F) \) of \( G \)
An Attempt

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If $F$ acyclic $\Rightarrow F$ is $s$-rooted spanning tree of $G$!
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Note: $F$ is optimum with respect to $c'$ (and therefore also with respect to $c$)
An Attempt

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An Attempt

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If $F$ acyclic $\Rightarrow$ $F$ is $s$-rooted spanning tree of $G$!

Note: $F$ is optimum with respect to $c'$ (and therefore also with respect to $c$) since $c'(F) = 0$. and $c' \geq 0$.

Problem: What if $F$ contains a cycle $K$?
Contraction

Consider (multi-) digraph $G = (V, E)$ and $U \subseteq V$ with edge costs $c : E \rightarrow \mathbb{R}_{\geq 0}$. 
Contraction

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Contraction of $U$: Replace $G[U]$ with new node $v_U$. 
Contraction

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**Contraction of $U$:** Replace $G[U]$ with new node $v_U$. 

![Diagram showing contraction](image)
Contraction

Consider (multi-) digraph $G = (V, E)$ and $U \subseteq V$ with edge costs $c : E \rightarrow \mathbb{R}_{\geq 0}$.

Contraction of $U$: Replace $G[U]$ with new node $v_U$. 

\[ G \xrightarrow{U} G/U \xrightarrow{\equiv} G/U \]
Contraction

Consider (multi-) digraph $G = (V, E)$ and $U \subseteq V$ with edge costs $c : E \to \mathbb{R}_{\geq 0}$.

**Contraction of $U$:** Replace $G[U]$ with new node $v_U$.

Edge costs are carried over to $G/U$. 

![Diagram of contraction](image)
**Expansion**

**Lem**². Let $K$ be a cycle in $F$ and $\tilde{T}$ an $s$-rooted spanning tree of $G/K$. Then there is an $s$-rooted spanning tree $T$ of $G$ with

$$c'(T) \leq c'(\tilde{T}).$$
Expansion

**Lem**². Let $K$ be a cycle in $F$ and $\tilde{T}$ an $s$-rooted spanning tree of $G/K$. Then there is an $s$-rooted spanning tree $T$ of $G$ with

$$c'(T) \leq c'(\tilde{T}).$$

**Proof.**

Every edge in $\tilde{T}$ corresponds to edge in $G$. 
Expansion

**Lem².** Let $K$ be a cycle in $F$ and $\tilde{T}$ an $s$-rooted spanning tree of $G/K$. Then there is an $s$-rooted spanning tree $T$ of $G$ with

$$c'(T) \leq c'(\tilde{T}).$$

**Proof.**

Every edge in $\tilde{T}$ corresponds to edge in $G$.

$\leadsto$ subgraph $H$ of $G$ with node set $V$. **Diagram:**

- $S$
- $u$
- $v$
- $v_K$
- $H$
Expansion

Lem\(^2\). Let \( K \) be a cycle in \( F \) and \( \tilde{T} \) an \( s \)-rooted spanning tree of \( G/K \). Then there is an \( s \)-rooted spanning tree \( T \) of \( G \) with

\[
c'(T) \leq c'({\tilde{T}}).
\]

Proof.

Every edge in \( \tilde{T} \) corresponds to edge in \( G \).

\( \Rightarrow \) subgraph \( H \) of \( G \) with node set \( V \).

Add cycle \( K \) to \( H \).
Expansion

**Lem**\(^2\). Let \( K \) be a cycle in \( F \) and \( \tilde{T} \) an \( s \)-rooted spanning tree of \( G/K \). Then there is an \( s \)-rooted spanning tree \( T \) of \( G \) with

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**Proof.**

Every edge in \( \tilde{T} \) corresponds to edge in \( G \).

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Expansion

Lem\(^2\). Let \(K\) be a cycle in \(F\) and \(\tilde{T}\) an \(s\)-rooted spanning tree of \(G/K\). Then there is an \(s\)-rooted spanning tree \(T\) of \(G\) with

\[ c'(T) \leq c' (\tilde{T}). \]

Proof.

Every edge in \(\tilde{T}\) corresponds to edge in \(G\).

\[ \leadsto \text{ subgraph } H \text{ of } G \text{ with node set } V. \]

Add cycle \(K\) to \(H\).

\[ \Rightarrow c'(H) = c'(\tilde{T}). \]
**Expansion**

**Lem**\(^2\). Let \( K \) be a cycle in \( F \) and \( \tilde{T} \) an \( s \)-rooted spanning tree of \( G/K \). Then there is an \( s \)-rooted spanning tree \( T \) of \( G \) with

\[ c'(T) \leq c' (\tilde{T}). \]

**Proof.**

Every edge in \( \tilde{T} \) corresponds to edge in \( G \).

\( \sim\sim \) subgraph \( H \) of \( G \) with node set \( V \).

Add cycle \( K \) to \( H \). \( \Rightarrow c'(H) = c'(\tilde{T}). \)

Every node in \( V \) is reachable in \( H \) from \( s \).
Expansion

\textbf{Lem}^2. Let $K$ be a cycle in $F$ and $\tilde{T}$ an $s$-rooted spanning tree of $G/K$. Then there is an $s$-rooted spanning tree $T$ of $G$ with

$$c'(T) \leq c'(\tilde{T}).$$

\textbf{Proof.}

Every edge in $\tilde{T}$ corresponds to edge in $G$. $\rightsquigarrow$ subgraph $H$ of $G$ with node set $V$.

Add cycle $K$ to $H$. $\Rightarrow c'(H) = c'(\tilde{T})$.

Every node in $V$ is reachable in $H$ from $s$.

Determine (arb.) $s$-rooted spanning tree $T$ of $H$. 
Expansion

**Lem**\(^2\). Let \(K\) be a cycle in \(F\) and \(\tilde{T}\) an \(s\)-rooted spanning tree of \(G/K\). Then there is an \(s\)-rooted spanning tree \(T\) of \(G\) with
\[
c'(T) \leq c'(\tilde{T}).
\]

**Proof.**

Every edge in \(\tilde{T}\) corresponds to edge in \(G\).
\(\leadsto\) subgraph \(H\) of \(G\) with node set \(V\).
Add cycle \(K\) to \(H\).
\(\Rightarrow c'(H) = c'(\tilde{T}).\)
Every node in \(V\) is reachable in \(H\) from \(s\).
Determine (arb.) \(s\)-rooted spanning tree \(T\) of \(H\).
\(T\) is \(s\)-rooted spanning tree of \(G\).
Expansion

**Lem**\(^2\). Let \(K\) be a cycle in \(F\) and \(\tilde{T}\) an \(s\)-rooted spanning tree of \(G/K\). Then there is an \(s\)-rooted spanning tree \(T\) of \(G\) with

\[
c'(T) \leq c'(\tilde{T}).
\]

**Proof.**

Every edge in \(\tilde{T}\) corresponds to edge in \(G\).

\(\sim\) subgraph \(H\) of \(G\) with node set \(V\).

Add cycle \(K\) to \(H\). \(\Rightarrow c'(H) = c'(\tilde{T})\).

Every node in \(V\) is reachable in \(H\) from \(s\).

Determine (arb.) \(s\)-rooted spanning tree \(T\) of \(H\).

\(T\) is \(s\)-rooted spanning tree of \(G\).

\[c'(T) \leq \]
Expansion

Lem\textsuperscript{2}. Let $K$ be a cycle in $F$ and $\tilde{T}$ an $s$-rooted spanning tree of $G/K$. Then there is an $s$-rooted spanning tree $T$ of $G$ with

$$c'(T) \leq c'(\tilde{T}).$$

Proof.

Every edge in $\tilde{T}$ corresponds to edge in $G$. 
\implies subgraph $H$ of $G$ with node set $V$.
Add cycle $K$ to $H$. 
\implies $c'(H) = c'(\tilde{T})$.

Every node in $V$ is reachable in $H$ from $s$.
Determine (arb.) $s$-rooted spanning tree $T$ of $H$. 
$T$ is $s$-rooted spanning tree of $G$.

$$c'(T) \leq c'(H) = c'(\tilde{T})$$
\hfill \Box
Algorithm

- Compute modified edge costs $c'$
Algorithm

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- Determine subgraph $F$
Algorithm

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- Determine subgraph $F$
- If $F$ is $s$-rooted spanning tree, return $F$
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- Otherwise determine cycle $K$ in $F$

[Edmonds 1967]
Algorithm

- Compute modified edge costs $c'$
- Determine subgraph $F$
- If $F$ is $s$-rooted spanning tree, return $F$
- Otherwise determine cycle $K$ in $F$
- Compute graph $G/K$

[Edmonds 1967]
Algorithm

- Compute modified edge costs $c'$
- Determine subgraph $F$
- If $F$ is $s$-rooted spanning tree, return $F$
- Otherwise determine cycle $K$ in $F$
- Compute graph $G/K$
- Apply algorithm recursively to $(G/K, c')$
  \[\rightsquigarrow\] $s$-rooted spanning tree $\tilde{T}$ for $G/K$

[Edmonds 1967]
Algorithm

1. Compute modified edge costs $c'$
2. Determine subgraph $F$
3. If $F$ is $s$-rooted spanning tree, return $F$
4. Otherwise determine cycle $K$ in $F$
5. Compute graph $G/K$
6. Apply algorithm recursively to $(G/K, c')$ resulting in $s$-rooted spanning tree $\tilde{T}$ for $G/K$
7. Expand $\tilde{T}$ to $s$-spanning tree $T$ of $G$ according to Lemma²
Algorithm

- Compute modified edge costs $c'$
- Determine subgraph $F$
- If $F$ is $s$-rooted spanning tree, return $F$
- Otherwise determine cycle $K$ in $F$
- Compute graph $G/K$
- Apply algorithm recursively to $(G/K, c')$ leading to $s$-rooted spanning tree $\tilde{T}$ for $G/K$
- Expand $\tilde{T}$ to $s$-spanning tree $T$ of $G$ according to Lemma$^2$
- Return $T$
Runtime

Algorithm terminates at the latest, when
Runtime

Algorithm terminates at the latest, when $|V| \leq 2$. 
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Algorithm terminates at the latest, when $|V| \leq 2$.

In every recursion stage the number of nodes is reduced by at least 1. $\Rightarrow$ $O(V)$ recursive calls.
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In every recursion stage the number of nodes is reduced by at least 1. $\Rightarrow O(V)$ recursive calls.

Cost modification, finding a cycle, contraction and expansion take $O(E)$ time each.
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Algorithm terminates at the latest, when $|V| \leq 2$.

In every recursion stage the number of nodes is reduced by at least 1. $\Rightarrow O(V)$ recursive calls.

Cost modification, finding a cycle, contraction and expansion take $O(E)$ time each.

**Thm.** Edmonds’ Algorithm terminates in $O(VE)$ time.
Optimality

**Lem**\(^3\). Let \( K \) be a cycle in \( F \) and \( T \) be an \( s \)-rooted spanning tree of \( G \). Then there is an \( s \)-rooted spanning tree \( \tilde{T} \) of \( G/K \) with \( c'(\tilde{T}) \leq c'(T) \).
Optimality

**Lem\(^2\).** Let \( K \) be a cycle in \( F \) and \( \tilde{T} \) be an \( s \)-rooted spanning tree of \( G/K \). Then there is an \( s \)-rooted spanning tree \( T \) of \( G \) with \( c'(T) \leq c'(\tilde{T}) \).

**Lem\(^3\).** Let \( K \) be a cycle in \( F \) and \( T \) be an \( s \)-rooted spanning tree of \( G \). Then there is an \( s \)-rooted spanning tree \( \tilde{T} \) of \( G/K \) with \( c'(\tilde{T}) \leq c'(T) \).
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Proof. Set \( H := T/K \).
Optimality

Lem\textsuperscript{3}. Let $K$ be a cycle in $F$ and $T$ be an $s$-rooted spanning tree of $G$. Then there is an $s$-rooted spanning tree $\tilde{T}$ of $G/K$ with $c'(\tilde{T}) \leq c'(T)$.

Proof. Set $H := T/K$.

$H$ is a subgraph of $G/K$ with $c'(H) \leq$
Optimality

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Proof. Set \( H := T/K \).

\( H \) is a subgraph of \( G/K \) with \( c'(H) \leq c'(T) \).
Optimality

**Lem**\(^3\). Let $K$ be a cycle in $F$ and $T$ be an $s$-rooted spanning tree of $G$. Then there is an $s$-rooted spanning tree $\tilde{T}$ of $G/K$ with $c'(\tilde{T}) \leq c'(T)$.

**Proof.** Set $H := T/K$.

$H$ ist subgraph of $G/K$ with $c'(H) \leq c'(T)$.

- Every $s$-$v$ path ($v \in V \setminus K$) in $T$ becomes a (not necessarily simple) $s$-$v$ path in $H$. 
Optimality

**Lem**\(^3\). Let \( K \) be a cycle in \( F \) and \( T \) be an \( s \)-rooted spanning tree of \( G \). Then there is an \( s \)-rooted spanning tree \( \tilde{T} \) of \( G/K \) with \( c'(\tilde{T}) \leq c'(T) \).

**Proof.** Set \( H := T/K \).

\( H \) is a subgraph of \( G/K \) with \( c'(H) \leq c'(T) \).

- Every \( s-v \) path \((v \in V \setminus K)\) in \( T \) becomes a (not necessarily simple) \( s-v \) path in \( H \).
- Every \( s-u \) path \((u \in K)\) in \( T \) becomes \( s-v_K \) path in \( H \).
Optimality

**Lem**$^3$. Let $K$ be a cycle in $F$ and $T$ be an $s$-rooted spanning tree of $G$. Then there is an $s$-rooted spanning tree $\tilde{T}$ of $G/K$ with $c'(\tilde{T}) \leq c'(T)$.

**Proof.** Set $H := T/K$.

$H$ is a subgraph of $G/K$ with $c'(H) \leq c'(T)$.

- Every $s-v$ path ($v \in V \setminus K$) in $T$ becomes a (not necessarily simple) $s-v$ path in $H$.
- Every $s-u$ path ($u \in K$) in $T$ becomes $s-v_K$ path in $H$.

$\Rightarrow$ Every node in $G/K$ is reachable in $H$ from $s$. 
Optimality

**Lem**³. Let $K$ be a cycle in $F$ and $T$ be an $s$-rooted spanning tree of $G$. Then there is an $s$-rooted spanning tree $\tilde{T}$ of $G/K$ with $c'(\tilde{T}) \leq c'(T)$.

**Proof.** Set $H := T/K$.

$H$ is subgraph of $G/K$ with $c'(H) \leq c'(T)$.

- Every $s$-$v$ path $(v \in V \setminus K)$ in $T$ becomes a (not necessarily simple) $s$-$v$ path in $H$.
- Every $s$-$u$ path $(u \in K)$ in $T$ becomes $s$-$v_K$ path in $H$.

$\Rightarrow$ Every node in $G/K$ is reachable in $H$ from $s$.

Consider (arbitrary) $s$-rooted spanning tree $\tilde{T}$ of $H$. 
Optimality

**Lem**\(^3\). Let \( K \) be a cycle in \( F \) and \( T \) be an \( s \)-rooted spanning tree of \( G \). Then there is an \( s \)-rooted spanning tree \( \tilde{T} \) of \( G/K \) with \( c'(\tilde{T}) \leq c'(T) \).

**Proof.** Set \( H := T/K \).

\( H \) is a subgraph of \( G/K \) with \( c'(H) \leq c'(T) \).

– Every \( s-v \) path \((v \in V \setminus K)\) in \( T \) becomes a (not necessarily simple) \( s-v \) path in \( H \).

– Every \( s-u \) path \((u \in K)\) in \( T \) becomes \( s-v_K \) path in \( H \).

\( \Rightarrow \) Every node in \( G/K \) is reachable in \( H \) from \( s \).

Consider (arbitrary) \( s \)-rooted spanning tree \( \tilde{T} \) of \( H \).

\( \Rightarrow \) \( \tilde{T} \) is also \( s \)-rooted spanning tree of \( G/K \).
Optimality

**Lem**³. Let $K$ be a cycle in $F$ and $T$ be an $s$-rooted spanning tree of $G$. Then there is an $s$-rooted spanning tree $\tilde{T}$ of $G\!\setminus\!K$ with $c'(\tilde{T}) \leq c'(T)$.

**Proof.** Set $H \coloneqq T\!\setminus\!K$.

$H$ is a subgraph of $G\!\setminus\!K$ with $c'(H) \leq c'(T)$.

- Every $s$-$v$ path ($v \in V \setminus K$) in $T$ becomes a (not necessarily simple) $s$-$v$ path in $H$.
- Every $s$-$u$ path ($u \in K$) in $T$ becomes $s$-$v_K$ path in $H$.

$\Rightarrow$ Every node in $G\!\setminus\!K$ is reachable in $H$ from $s$.

Consider (arbitrary) $s$-rooted spanning tree $\tilde{T}$ of $H$.

$\Rightarrow \tilde{T}$ is also $s$-rooted spanning tree of $G\!\setminus\!K$ and we have $c'(\tilde{T}) \leq \ldots$
Optimality

Lem\ref{lem:optimality}. Let $K$ be a cycle in $F$ and $T$ be an $s$-rooted spanning tree of $G$. Then there is an $s$-rooted spanning tree $\tilde{T}$ of $G/K$ with $c'(\tilde{T}) \leq c'(T)$.

Proof. Set $H := T/K$.

$H$ is a subgraph of $G/K$ with $c'(H) \leq c'(T)$.

– Every $s$-$v$ path ($v \in V \setminus K$) in $T$ becomes a (not necessarily simple) $s$-$v$ path in $H$.

– Every $s$-$u$ path ($u \in K$) in $T$ becomes $s$-$v_K$ path in $H$.

$\Rightarrow$ Every node in $G/K$ is reachable in $H$ from $s$.

Consider (arbitrary) $s$-rooted spanning tree $\tilde{T}$ of $H$.

$\Rightarrow \tilde{T}$ is also $s$-rooted spanning tree of $G/K$ and we have $c'(\tilde{T}) \leq c'(H) \leq c'(T)$.
Optimality

**Lemma 3.** Let $K$ be a cycle in $F$ and $T$ be an $s$-rooted spanning tree of $G$. Then there is an $s$-rooted spanning tree $\tilde{T}$ of $G/K$ with $c'(\tilde{T}) \leq c'(T)$.

**Proof.** Set $H := T/K$.

$H$ is a subgraph of $G/K$ with $c'(H) \leq c'(T)$.

- Every $s-v$ path ($v \in V \setminus K$) in $T$ becomes a (not necessarily simple) $s-v$ path in $H$.
- Every $s-u$ path ($u \in K$) in $T$ becomes $s-v_K$ path in $H$.

$\Rightarrow$ Every node in $G/K$ is reachable in $H$ from $s$.

Consider (arbitrary) $s$-rooted spanning tree $\tilde{T}$ of $H$.

$\Rightarrow \tilde{T}$ is also $s$-rooted spanning tree of $G/K$ and we have $c'(\tilde{T}) \leq c'(H) \leq c'(T)$. \qed
Optimality

**Thm.** Edmonds’ Algorithm computes a minimum $s$-rooted spanning tree of $G$. 
Optimality

**Thm.** Edmonds’ Algorithm computes a minimum $s$-rooted spanning tree of $G$.

**Proof.**
By induction on the number of nodes.
If $F$ is acyclic, the algorithm is correct.
Optimality

**Thm.** Edmonds’ Algorithm computes a minimum s-rooted spanning tree of $G$.

**Proof.**

By induction on the number of nodes.

If $F$ is acyclic, the algorithm is correct.

Let $T^*$ minimum s-rooted spanning tree of $G$ and $K$ a cycle in $F$
Optimality

**Thm.** Edmonds’ Algorithm computes a minimum s-rooted spanning tree of $G$.

**Proof.**
By induction on the number of nodes.
If $F$ is acyclic, the algorithm is correct.
Let $T^*$ minimum s-rooted spanning tree of $G$ and $K$ a cycle in $F$.

Lemma$^3$ $\Rightarrow$ exist. s-RST $\tilde{T}$ of $G/K$ with $c'(\tilde{T}) \leq c'(T^*)$.

$= : \text{OPT'}$
Optimality

**Thm.** Edmonds’ Algorithm computes a minimum $s$-rooted spanning tree of $G$.

**Proof.**
By induction on the number of nodes.
If $F$ is acyclic, the algorithm is correct.
Let $T^*$ minimum $s$-rooted spanning tree of $G$ and $K$ a cycle in $F$.
Lemma$^3 \Rightarrow$ exist. $s$-RST $\tilde{T}$ of $G/K$ with $c'(\tilde{T}) \leq c'(T^*)$.
Inductive hypothesis $\Rightarrow$
Algorithm yields $s$-RST $\hat{T}$ of $G/K$ with $c'(\hat{T}) \leq c'(\tilde{T})$. $=: \text{OPT'}$
Optimality

**Thm.** Edmonds’ Algorithm computes a minimum s-rooted spanning tree of $G$.

**Proof.**

By induction on the number of nodes.

If $F$ is acyclic, the algorithm is correct.

Let $T^*$ minimum s-rooted spanning tree of $G$ and $K$ a cycle in $F$.

Lemma\(^3\) $\Rightarrow$ exist. s-RST $\tilde{T}$ of $G/K$ with $c'(\tilde{T}) \leq c'(T^*)$.

Inductive hypothesis $\Rightarrow$

Algorithm yields s-RST $\hat{T}$ of $G/K$ with $c'(\hat{T}) \leq c'(\tilde{T})$.

Lemma\(^2\) yields s-RST $T$ of $G$ with $c'(T) \leq c'(\hat{T}) \leq c'(T^*)$.

$=: \text{OPT}'$
Optimality

**Thm.** Edmonds’ Algorithm computes a minimum s-rooted spanning tree of \( G \).

**Proof.**

By induction on the number of nodes.

If \( F \) is acyclic, the algorithm is correct.

Let \( T^* \) minimum s-rooted spanning tree of \( G \) and \( K \) a cycle in \( F \).

Lemma\(^3\) \( \Rightarrow \) exist. s-RST \( \tilde{T} \) of \( G/K \) with \( c'(\tilde{T}) \leq c'(T^*) \).

Inductive hypothesis \( \Rightarrow \)

Algorithm yields s-RST \( \hat{T} \) of \( G/K \) with \( c'(\hat{T}) \leq c'(\tilde{T}) \).

Lemma\(^2\) yields s-RST \( T \) of \( G \) with \( c'(T) \leq c'(\hat{T}) \leq c'(T^*) \).

\( \Rightarrow \) \( T \) is optimum w.r.t. \( c' \) and thus w.r.t. \( c \).

\[ \boxed{\text{OPT}' = \text{OPT}'} \]